The Modern Form of Bernoulli's Theorem

The modern form of Bernoulli's theorem is:

If N is sufficiently large, the probability that $\left|\frac{M}{N}-P\right| < \varepsilon$ will be greater than 1 - η . N $-P \leq \varepsilon$ will be greater than 1

M is the number of successes in N trials, P is the probability of a success in a single trial and ε and η are positive numbers chosen as small as desired.

In this paper, I outline the proof of this theorem using Bernoulli's and Markov's techniques.

One of the improvements that Markov made to Bernoulli's theorem was to remove the restrictions that Bernoulli put on N and ϵ .

In Bernoulli's proof ε had to be $\frac{1}{\pi}$ and N had to be a multiple of T. T Bernoulli used NT to stand for the number of trials. I will use N* to

stand for Bernoulli's use of N and N will stand for the number of trials.

 NP roughly corresponds to N^*R

 $NP + N \varepsilon$ roughly corresponds to $N^*R + N^*$

 $NP - N \varepsilon$ roughly corresponds to $N^*R - N^*$

I say roughly because NP, $NP + N \varepsilon$, and NP - N ε may not be integers.

Let λ be the smallest integer greater than or equal to NP. Let μ be the smallest integer greater than or equal to NP + N ϵ . Let k be the largest integer less than or equal to $NP - N \varepsilon$. Let T_i be the probability of getting exactly i successes in N trials.

The probability that
$$
\left| \frac{M}{N} - P \right| < \varepsilon
$$
 is equal to:
\n
$$
T_{k+1} + T_{k+2} + \dots + T_{\lambda} + T_{\lambda+1} + \dots + T_{\mu-1}
$$
\nand the probability that\n
$$
\left| \frac{M}{N} - P \right| \geq \varepsilon
$$
\nis equal to: $T_0 + T_1 + \dots + T_k + T_{\mu} + T_{\mu+1} + \dots + T_N$ \nLet $A = T_{\lambda} + T_{\lambda+1} + \dots + T_{\mu-1}$ and $B = T_{\lambda-1} + T_{\lambda-2} + \dots + T_{k+1}$ \nLet $C = T_{\mu} + T_{\mu+1} + \dots + T_N$ and $D = T_k + T_{k-1} + \dots + T_0$ \n\nThe probability that $\left| \frac{M}{N} - P \right| < \varepsilon$ is $A + B$ and $A + B + C + D = 1$.

 $\left| \begin{array}{cc} \text{U} & \text{V} \\ \text{N} & \text{V} \end{array} \right|$ We will show that if N is sufficiently large, $C < \frac{A\eta}{1}$ and $D < \frac{B\eta}{1}$ $1-\eta$ $1-\eta$ $B\eta$ $1-\eta$

So then
$$
A+B + \frac{A\eta}{1-\eta} + \frac{B\eta}{1-\eta} > 1
$$
. So $A+B > 1 - \eta$.

So if N is sufficiently large, the probability that $\left|\frac{M}{N} - P\right| < \varepsilon$ is N $-P$ $<$ ϵ is greater than $1 - \eta$.

$$
\frac{T_{\mu}}{T_{\lambda}} > \frac{T_{\mu+1}}{T_{\lambda+1}} > \frac{T_{\mu+2}}{T_{\lambda+2}} > \dots > \frac{T_{N}}{T_{N-(\mu-\lambda)}}
$$
\n
$$
\text{Let } A_1 = \Gamma_{\mu} + \Gamma_{\mu+1} + \dots + \Gamma_{2\mu-\lambda+1}
$$
\n
$$
A_2 = \Gamma_{2\mu-2} + \Gamma_{2\mu-2\lambda+1} + \dots + \Gamma_{3\mu-2\lambda-1}
$$
\n
$$
A_3 = \Gamma_{3\mu-2\lambda} + \Gamma_{3\mu-2\lambda+1} + \dots + \Gamma_{4\mu-3\lambda-1}
$$
\n
$$
\vdots
$$
\n
$$
\frac{T_{\mu}}{T_{\lambda}} > \frac{A_1}{A}, \quad \frac{T_{\mu}}{T_{\lambda}} > \frac{A_2}{A_1}, \quad \frac{T_{\mu}}{T_{\lambda}} > \frac{A_3}{A_2}, \quad \frac{T_{\mu}}{T_{\lambda}} > \frac{A_4}{A_3}, \dots
$$
\n
$$
\text{So } \left(\frac{T_{\mu}}{T_{\lambda}}\right)^2 > \frac{A_2}{A}, \left(\frac{T_{\mu}}{T_{\lambda}}\right)^3 > \frac{A_2}{A}, \left(\frac{T_{\mu}}{T_{\lambda}}\right)^4 > \frac{A_4}{A}, \dots
$$
\n
$$
\text{So } \frac{T_{\mu}}{T_{\lambda}} + \left(\frac{T_{\mu}}{T_{\lambda}}\right)^2 + \left(\frac{T_{\mu}}{T_{\lambda}}\right)^3 + \left(\frac{T_{\mu}}{T_{\lambda}}\right)^4 + \dots > \frac{A_1 + A_2 + A_3 + \dots}{A}
$$
\n
$$
\text{So } \frac{T_{\mu}}{T_{\lambda}} + \left(\frac{T_{\mu}}{T_{\lambda}}\right)^2 + \left(\frac{T_{\mu}}{T_{\lambda}}\right)^3 + \left(\frac{T_{\mu}}{T_{\lambda}}\right)^4 + \dots > \frac{T_{\mu} + T_{\mu+1} + \dots + T_N}{A}
$$
\n
$$
\text{So } \frac{T_{\mu}}{T_{\lambda}} > \frac{T_{\mu} + T_{\mu+1} + \dots + T_N}{A} = \frac{C}{A}. \quad \text{So } A \begin{pmatrix} \frac{T_{\mu}}{
$$

So
$$
\frac{T_{\mu}}{T_{\lambda}} > \frac{T_{\mu} + T_{\mu+1} + \dots + T_N}{A} = \frac{C}{A}. \quad \text{So A } \left(\frac{\frac{T_{\mu}}{T_{\lambda}}}{1 - \frac{T_{\mu}}{T_{\lambda}}}\right) > C
$$

\n- \n Since
$$
k < \lambda - 1
$$
, $\frac{T_k}{T_{\lambda-1}} > \frac{T_{k-1}}{T_{\lambda-2}} > \frac{T_{k-2}}{T_{\lambda-3}} > \dots > \frac{T_0}{T_{\lambda-k-1}}$ \n
\n- \n Let $B_1 = T_{1} + T_{k-1} + \dots + T_{2k-1}$.\n
\n

• Let
$$
B_1 = T_k + T_{k-1} + \dots + T_{2k-\lambda+2}
$$

$$
B_2 = T_{2\kappa-\lambda+1} + T_{2\kappa-\lambda} + \dots + T_{3\kappa-2\lambda+3}
$$

\n
$$
B_3 = T_{3\kappa-2\lambda+2} + T_{3k-2\lambda+1} + \dots + T_{4k-3\lambda+4}
$$

\n*

$$
B_{3} = T_{3k-2\lambda+2} + T_{3k-2\lambda+1} + \dots + T_{4k-3\lambda+4}
$$
\n*
\n*
\n*
\n*
\n*
\n
$$
\frac{T_{k}}{T_{k-1}} > \frac{B_{1}}{B}, \quad \frac{T_{k}}{T_{k-1}} > \frac{B_{2}}{B_{1}}, \quad \frac{T_{k}}{T_{k-1}} > \frac{B_{3}}{B_{2}} \dots
$$
\n
$$
\frac{T_{k}}{T_{k-1}} > \frac{B_{1}}{B}, \quad \left(\frac{T_{k}}{T_{k-1}}\right)^{2} > \frac{B_{2}}{B}, \quad \left(\frac{T_{k}}{T_{k-1}}\right)^{3} > \frac{B_{3}}{B} \dots
$$
\n
$$
S_{0} \quad \frac{T_{k}}{T_{k-1}} + \left(\frac{T_{k}}{T_{k-1}}\right)^{2} + \left(\frac{T_{k}}{T_{k-1}}\right)^{3} + \dots > \frac{B_{1} + B_{2} + B_{3} + \dots}{B}
$$
\n
$$
S_{0} \quad \frac{T_{k}}{T_{k-1}} + \left(\frac{T_{k}}{T_{k-1}}\right)^{2} + \left(\frac{T_{k}}{T_{k-1}}\right)^{3} + \dots > \frac{T_{k} + T_{k-1} + \dots + T_{0}}{B} = \frac{D}{B}
$$

B

So
$$
\frac{\frac{T_k}{T_{\lambda-1}}}{1-\frac{T_k}{T_{\lambda-1}}} > \frac{D}{B}
$$
, So $B \frac{\frac{T_k}{T_{\lambda-1}}}{1-\frac{T_k}{T_{\lambda-1}}} > D$

It remains to be shown that by making N sufficiently large

$$
\frac{T_{\mu}}{T_{\lambda}}
$$
 nd $\frac{T_{k}}{T_{\lambda-1}}$ can be made smaller than η .

We will show that $\frac{T_{\lambda}}{T_{\lambda}}$ And $\frac{T_{\lambda-1}}{T_{\lambda}}$ can be made as large as T_{μ} T_{k} \cdots T_{k} λ And λ - con he mode μ κ T_{i-1} can be made as large a T_k . There is the set of \mathcal{L}_k λ -1 - can be made as large as desired by making N sufficently large, which gives the same result. the as large as

the gives the same

my paper
 $\frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$

tly K successes in de as large as

h gives the same

my paper
 $\frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$

ttly K successes in

single trial is P x as large as
gives the same
apper
 $\frac{P^{K}Q^{N-K}}{P(N-K)!}$
v K successes in Example 23

gives the same

example 19
 $\frac{P^K Q^{N-K}}{P(N-K)!}$

(N K successes in mgle trial is P

 We will apply my method from Lemma 7 of my paper Bernoulli's Theorem.

We calculate
$$
\frac{T_{\lambda}}{T_{\mu}}
$$
 from the formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$

Where $P(K)$ is the probability of getting exactly K successes in N trials when the probability of success on a single trial is P and $Q = 1 - P$. be will apply my method from Lemma 7 of my paper

ernoulli's Theorem.

Ve calculate $\frac{T_i}{T_n}$ from the formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$

There P(K) is the probability of getting exactly K successes in

trials when the Il apply my method from Lemma 7 of my paper

ulli's Theorem.

culate $\frac{T_s}{T_n}$ from the formula $P(K) = \frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$
 $P(K)$ is the probability of getting exactly K successes in

when the probability of success on my method from Lemma 7 of my paper

<u>eorem.</u>
 $\frac{T_{\lambda}}{T_{\mu}}$ from the formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$

the probability of getting exactly K successes in

the probability of success on a single trial is P
 $\frac{(N - \lambda + 1$ pply my method from Lemma 7 of my paper

's Theorem.

late $\frac{T_{\lambda}}{T_{\mu}}$ from the formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$

K) is the probability of getting exactly K successes in

nen the probability of success on a single ply my method from Lemma 7 of my paper

Theorem.

te $\frac{T_{\lambda}}{T_{\mu}}$ from the formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$

) is the probability of getting exactly K successes in

in the probability of success on a single trial is method from Lemma 7 of my paper

<u>m.</u>

from the formula $P(K) = \frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$

probability of getting exactly K successes in

robability of success on a single trial is P
 $-\lambda + 1) P^{\lambda} Q^{N-\lambda}$

1)......1 y my method from Lemma 7 of my paper

<u>heorem.</u>
 $\frac{T_{\lambda}}{T_{\mu}}$ from the formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$

is the probability of getting exactly K successes in

the probability of success on a single trial is P

....from Lemma 7 of my paper
formula $P(K) = \frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$
lity of getting exactly K successes in
ty of success on a single trial is P
 λQ^{N-A}

$$
T_{\lambda} = \frac{N(N-1)...(N-\lambda+1)P^{\lambda}Q^{N-\lambda}}{\lambda(\lambda-1)...1}
$$

$$
T_{\mu} = \frac{N(N-1)...(N-\mu+1)P^{\mu}Q^{N-\mu}}{\mu(\mu-1)...1}
$$

$$
T_{\mu} = \frac{N(N-1)...(N-\mu+1)P^{\mu}Q^{N-\mu}}{\mu(\mu-1)...1}
$$

So
$$
\frac{T_{\lambda}}{T_{\mu}} = \frac{\mu(\mu-1)...(\lambda+1)Q^{\mu-\lambda}}{(N-\lambda)(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}}
$$

Reversing the order of the factors in both the numerator and

Reversing the order of the factors in both the numerator and denominator so they will be increasing from left to right instead of decreasing we get: $\mu(\mu-1)$($\lambda+1$) $Q^{\mu-\lambda}$
 λ)($N-\lambda-1$)....($N-\mu+1$) $P^{\mu-\lambda}$

der of the factors in both the numerator and

they will be increasing from left to right instead of

et:

($\lambda+1$)($\lambda+2$)....($\mu-1$) $\mu Q^{\mu-\lambda}$

)(= $\frac{\mu(\mu-1)...(\lambda+1)Q^{\mu-\lambda}}{(N-\lambda)(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}}$

ng the order of the factors in both the numerator and

nator so they will be increasing from left to right instead of

ing we get:
 $\frac{(\lambda+1)(\lambda+2)...(\mu-1)\mu Q^{\mu-\lambda}}{(N-\mu+1)(N-\mu$ $\mu(\mu-1)...(2+1)Q^{\mu-\lambda}$
 $\mu(\mu-1)...(2+1)Q^{\mu-\lambda}$
 $\lambda(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}$

der of the factors in both the numerator and

they will be increasing from left to right instead of

t:
 $\lambda+1(2+2)...(\mu-1)\mu Q^{\mu-\lambda}$
 $(N-\mu+2)...(N-\lambda-1)(N-\lambda)$ $\mu(\mu-1)...(\lambda+1)Q^{\mu-\lambda}$
 $N-\lambda(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}$

are order of the factors in both the numerator and

so they will be increasing from left to right instead of

ve get:
 $(\lambda+1)(\lambda+2)...(\mu-1)\mu Q^{\mu-\lambda}$
 $(\mu+1)(N-\mu+2)...(N-\lambda-1)(N-\lambda)P^$ $\sqrt{p^{\mu-\lambda}}$
oth the numerator and
from left to right instead of
 $\frac{p^{\mu-\lambda}}{(N-\lambda)P^{\mu-\lambda}}$ umerator and
ft to right instead of $\mu(\mu-1)...(1+\mu)^{n-\lambda}$
 $\mu(\mu-1)...(2+\mu)^{n-\lambda}$
 $N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}$

ar of the factors in both the numerator and
 μ will be increasing from left to right instead of
 $+1$)($\lambda + 2)....(\mu - 1) \mu Q^{\mu-\lambda}$
 $N-\mu+2)....(N-\lambda-1)(N-\lambda)P^{\$ $\frac{\mu(\mu-1)...(\lambda+1)Q^{\mu-\lambda}}{(N-\lambda)(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}}$
the order of the factors in both the numerator and
or so they will be increasing from left to right instead of
we get:
 $\frac{(\lambda+1)(\lambda+2)...(\mu-1)\mu Q^{\mu-\lambda}}{-\mu+1)(N-\mu+2)...(N-\lambda-1)(N-\lambda)P^{\$ $u(\mu-1)$($\lambda + 1$) $Q^{\mu-\lambda}$
 $(\lambda - \lambda - 1)$($N - \mu + 1$) $P^{\mu-\lambda}$

of the factors in both the numerator and

will be increasing from left to right instead of
 1)($\lambda + 2$)....($\mu - 1$) $\mu Q^{\mu-\lambda}$
 $-\mu + 2$)....($N - \lambda -$ $\mu(\mu-1)\dots(\lambda+1)Q^{\mu-\lambda}$
 $\lambda(N-\lambda-1)\dots(N-\mu+1)P^{\mu-\lambda}$

rder of the factors in both the numerator and

they will be increasing from left to right instead of

get:
 $(\lambda+1)(\lambda+2)\dots(\mu-1)\mu Q^{\mu-\lambda}$
 $1)(N-\mu+2)\dots(N-\lambda-1)(N-\lambda)P^{\mu-\lambda}$
 $\qquad \$ = $\frac{\mu(\mu-1)...(\lambda+1)Q^{\mu-\lambda}}{(N-\lambda)(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}}$

ag the order of the factors in both the numerator and

ator so they will be increasing from left to right instead of
 $\frac{(\lambda+1)(\lambda+2)...(\mu-1)\mu Q^{\mu-\lambda}}{N-\mu+1)(N-\mu+2)...(N-\lambda-1)(N-\lambda)$ g the order of the factors in both the numerator and

g the order of the factors in both the numerator and

ator so they will be increasing from left to right instead

ng we get:
 $(\lambda + 1)(\lambda + 2)...(\mu - 1)\mu Q^{\mu-\lambda}$
 $N - \mu + 1)(N$ sing the order of the factors in both the numerator and

sing we get:
 $\frac{(\lambda + 1)(\lambda + 2) \dots (\mu - 1)\mu Q^{\mu - \lambda}}{(N - \mu + 1)(N - \mu + 2) \dots (N - \lambda - 1)(N - \lambda)P^{\mu - \lambda}}$
 $\frac{(\lambda Q + Q)}{(NP - \mu P + P)} * \frac{(\lambda Q + 2Q)}{(NP - \mu P + 2P)} * ... * \frac{(\mu Q - Q)}{(NP - \lambda P - P)} * \frac{\mu Q}{(NP \frac{1}{(N-\lambda)(N-\lambda-1)...(N-\mu+1)P^{\mu-\lambda}}$
the order of the factors in both the numerator and
or so they will be increasing from left to right instea
we get:
 $\frac{(\lambda+1)(\lambda+2)...(\mu-1)\mu Q^{\mu-\lambda}}{-\mu+1)(N-\mu+2)...(N-\lambda-1)(N-\lambda)P^{\mu-\lambda}}$
 $\frac{Q+Q}{(\lambda P-\mu P$ ing the order of the factors in both the numerator and

inator so they will be increasing from left to right instead of

ing we get:
 $(\lambda + 1)(\lambda + 2)....(\mu - 1)\mu Q^{\mu-\lambda}$
 $\frac{(\lambda + 1)(\lambda + 2)...(\lambda - \lambda - 1)(\lambda - \lambda)P^{\mu-\lambda}}{(\lambda - \mu + 1)(N - \mu + 2)...$ the order of the factors in both the numerator and
the order of the factors in both the numerator and
is we get:
 $(\lambda + 1)(\lambda + 2)...(\mu - 1)\mu Q^{\mu-\lambda}$
 $-\mu + 1)(N - \mu + 2)...(N - \lambda - 1)(N - \lambda)P^{\mu-\lambda}$
 $\frac{Q + Q}{\mu P + P} * \frac{(\lambda Q + 2Q)}{(NP - \mu P + 2P)} * * \$ $\frac{1}{(1-1)...(N-\mu+1)P^{\mu-\lambda}}$

the factors in both the numerator and

II be increasing from left to right instead of
 $x+2)...(u-1)\mu Q^{\mu-\lambda}$
 $x+2)...(N-\lambda-1)(N-\lambda)P^{\mu-\lambda}$
 $\frac{(\lambda Q+2Q)}{(N-\lambda P-\mu P)}$ $\frac{\mu Q}{(NP-\lambda P-\mu P)}$
 $\frac{\mu Q}{(NP-\lambda P-\mu P)}$
 of the factors in both the numerator and

will be increasing from left to right instead of
 $((\lambda + 2)...(\mu - 1)\mu Q^{\mu-\lambda})$
 $(\mu + 2)...(N - \lambda - 1)(N - \lambda)P^{\mu-\lambda}$
 $(\lambda Q + 2Q)$
 $(NP - \mu P + 2P) * ... * (\mu Q - Q) * \mu Q$
 $(NP - \lambda P - P) * (\lambda P - \lambda P)$

tion is obt 2. $(N-\mu+1)P^{\mu-\lambda}$

2. factors in both the numerator and

be increasing from left to right instead of

2..... $(\mu-1)\mu Q^{\mu-\lambda}$

2..... $(N-\lambda-1)(N-\lambda)P^{\mu-\lambda}$
 $Q+2Q$
 $\mu P+2P$ ** $\frac{(\mu Q-Q)}{(NP-\lambda P-P)}$ * $\frac{\mu Q}{(NP-\lambda P)}$

is obta F the factors in both the numerator and

vill be increasing from left to right instead of
 $(\lambda + 2)....(\mu - 1)\mu Q^{\mu-\lambda}$
 $(\mu + 2)....(N - \lambda - 1)(N - \lambda)P^{\mu-\lambda}$
 $\frac{(\lambda Q + 2Q)}{NP - \mu^p + 2P}$ ** $\frac{(\mu Q - Q)}{(NP - \lambda P - P)}$ * $\frac{\mu Q}{(NP - \lambda P)}$
 e factors in both the numerator and

be increasing from left to right instead of
 $2)....(u-1)\mu Q^{u-\lambda}$
 $2)....(N-\lambda-1)(N-\lambda)P^{u-\lambda}$
 $\frac{(Q+2Q)}{-\mu^2+2P}$ * * $\frac{(\mu Q-Q)}{(NP-\lambda P-P)}$ * $\frac{\mu Q}{(NP-\lambda P)}$

is obtained from the previous f : numerator and

left to right instead of
 $(\mu Q - Q)$
 $(\mu Q - Q)$
 $(\mu Q - P)$
 $(\mu Q - P)$
 $(\mu Q - \lambda P)$ the numerator and

m left to right instead of
 $(\mu Q - Q)$
 $(\mu Q - Q)$
 $(\mu P - \lambda P - P)$
 $(\mu P - \lambda P)$

the previous fraction by

mominator. umerator and

ft to right instead of
 $\frac{Q-Q}{\sqrt{P-P}}$ * $\frac{\mu Q}{(NP-AP)}$

previous fraction by the numerator and
 $\lambda P^{\mu-\lambda}$
 $(\mu Q - Q)$
 $(NP - \lambda P - P)$
 $(NP - \lambda P)$

the previous fraction by

nominator. umerator and

ff to right instead of
 $\frac{Q-Q}{\sqrt{Q^2-Q^2}}$
 $\frac{Q-Q}{(NP-AP)}$

previous fraction by a

instead of
 $\frac{\mu Q}{(NP - \lambda P)}$

action by

$$
\frac{T_{\lambda}}{T_{\mu}} = \frac{(\lambda+1)(\lambda+2)...(\mu-1)\mu Q^{\mu-\lambda}}{(N-\mu+1)(N-\mu+2)...(N-\lambda-1)(N-\lambda)P^{\mu-\lambda}}
$$

$$
\frac{T_{\lambda}}{T_{\mu}} = \frac{(\lambda Q + Q)}{(NP - \mu P + P)} * \frac{(\lambda Q + 2Q)}{(NP - \mu P + 2P)} * \dots * \frac{(\mu Q - Q)}{(NP - \lambda P - P)} * \frac{\mu Q}{(NP - \lambda P)}
$$

 Notice that each fraction is obtained from the previous fraction by adding Q to the numerator and P to the denominator .

The first fraction is itself obtained from $\frac{\lambda Q}{\lambda P}$ by adding Q to the Q by adding Q to the $\frac{(\mu Q - Q)}{(NP - \lambda P - P)} * \frac{\mu Q}{(NP - \lambda P)}$

in the previous fraction by

enominator.
 $\frac{\lambda Q}{NP - \mu P}$ by adding Q to the

re are $\mu - \lambda$ fractions in the

emma 7, $\frac{T_{\lambda}}{T_{\lambda}}$ is numerator and P to the denominator. There are $\mu-\lambda$ fractions in the product, so by the same reasoning as in lemma 7, $\frac{T_{\lambda}}{T_{\lambda}}$ is T_{μ} λ is a set of λ μ

greater than the smaller of $\left(\frac{\lambda Q}{\lambda R}\right)^{\mu-\lambda}$ or $\left(\frac{\mu Q}{\lambda R}\right)^{\mu-\lambda}$. μ $\left(\frac{N}{P} - \frac{\lambda P}{P}\right)$ (λQ) $\mu-\lambda$ or (μQ) $\mu-\lambda$ $\frac{\lambda Q}{NP - \mu P}$ $\mu \rightarrow \lambda$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right) \mu \rightarrow \lambda$.

ers then $\mu \rightarrow \lambda = N\epsilon$. If one or both are
 -1 . $\langle NP - \mu P \rangle$ $\langle (\Lambda$ $\left| \frac{R_{\Sigma}}{M} \right|$ \sim Or $\left| \frac{R}{M} \right|$ $\mu-\lambda$ or μQ | $\left(\frac{(NP - \lambda P)}{P}\right)$ $\int^{\mu-\lambda}$ or $\left(\frac{\mu Q}{(NP-\lambda P)}\right)^{\mu-\lambda}$. $(\mu Q)_{\mu-\lambda}$ $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{\mu - \lambda}$.

NE. If one or both are $\left(\frac{PZ}{(NP - \lambda P)}\right)^{\mu \alpha}$. $\mu-\lambda$ \int ^{μ} \ldots If NP and N ε are both integers then $\mu-\lambda=N\varepsilon$. If one or both are not integers, then $\mu-\lambda > N\epsilon - 1$. So $\frac{T_{\lambda}}{T_{\lambda}}$ is greater than the smaller of $\left(\frac{\lambda Q}{\lambda W_{\lambda}}\right)^{N_{\epsilon}-1}$ or $\left(\frac{\mu Q}{\lambda W_{\lambda}}\right)^{N_{\epsilon}-1}$. T_{μ} λ is cusatom than the small μ λO \vert $N_{\epsilon-1}$ \vert μ \vert \vert $N_{\epsilon-1}$ μ $\left(\frac{N}{P} - \lambda P\right)$ (λQ) $N_{\epsilon-1}$ αr (μQ) $N_{\epsilon-1}$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{\mu - \lambda}$.
 $\mathbf{L} - \lambda = \mathbf{N} \varepsilon$. If one or both are
 $\frac{\lambda Q}{NP - \mu P}\right)^{N\varepsilon - 1}$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{N\varepsilon - 1}$.
 $= \frac{P + \varepsilon}{P}$. $\langle NP - \mu P \rangle$ $\langle l \rangle$ $\left(\frac{n_{\mathcal{L}}}{\sqrt{n_{\mathcal{L}}}}\right)^{1/2}$ or $\left(\frac{n_{\mathcal{L}}}{\sqrt{n_{\mathcal{L}}}}\right)$ $N\varepsilon-1$ or μQ) $\left(\frac{(NP - \lambda P)}{\lambda}\right)$ $\int^{N\epsilon-1}$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{N\epsilon-1}$. (μQ) $N_{\epsilon-1}$ λ .

e or both are
 $\frac{\mu Q}{(NP - \lambda P)}$ ^{N_{E-1}}. $\left(\frac{PZ}{(NP - \lambda P)}\right)^{1/3}$. $N\epsilon-1$ \int $\frac{1}{2}$ $\frac{1}{$ greater than the smaller of $\left(\frac{\lambda Q}{NP - \mu P}\right)^{\mu-\lambda}$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{\mu-\lambda}$.

If NP and Nε are both integers then $\mu-\lambda = \text{N}\epsilon$. If one or

not integers, then $\mu-\lambda > \text{N}\epsilon - 1$.

So $\frac{T_{\lambda}}{T_{\mu}}$ is greater than the $\frac{\lambda Q}{NP - \mu P}$)^{$\mu \rightarrow \lambda$} or $\left(\frac{\mu Q}{(NP - \lambda P)}\right) \mu \rightarrow \lambda$.

ers then $\mu \rightarrow \lambda = N\epsilon$. If one or both are

-1.

naller of $\left(\frac{\lambda Q}{NP - \mu P}\right)$ ^{Ne-1} or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)$ ^{Ne-1}.
 $\frac{(NP + Ne)Q}{NP - NP^2} = \frac{P + \epsilon}{P}$.

nd $\frac{NPQ}{P} = \frac{Q$ greater than the smaller of $\left(\frac{\lambda Q}{NP - \mu P}\right)^{\mu-\lambda}$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{\mu-\lambda}$.

f NP and Ne are both integers then $\mu-\lambda = \text{N}\varepsilon$. If one or both are

not integers, then $\mu-\lambda > \text{N}\varepsilon - 1$.

So $\frac{T_{\lambda}}{T_{\mu}}$ is grea $\left(\frac{\mu}{(NP - \lambda P)}\right)^{\mu - \lambda}$ or $\left(\frac{\mu}{(NP - \lambda P)}\right)^{\mu - \lambda}$.

then $\mu - \lambda = N\varepsilon$. If one or both are
 $\left(\frac{\lambda Q}{NP - \mu P}\right)^{N\varepsilon - 1}$ or $\left(\frac{\mu Q}{(NP - \lambda P)}\right)^{N\varepsilon - 1}$.
 $\frac{N\rho Q}{(NP - \lambda P)^2} = \frac{P + \varepsilon}{P}$.
 $\frac{NPQ}{NP - (NP + N\varepsilon)P} = \frac{Q}{Q -$

$$
\frac{(NP + N\varepsilon)Q}{NP - NP^2} \le \frac{\mu Q}{NP - \lambda P} \quad \text{and} \quad \frac{(NP + N\varepsilon)Q}{NP - NP^2} = \frac{P + \varepsilon}{P}.
$$
\n
$$
NPQ \qquad \qquad \lambda Q \qquad \text{and} \qquad NPQ \qquad \qquad Q
$$

$$
\frac{NPQ}{NP - (NP + N\varepsilon)P} \le \frac{\lambda Q}{NP - \mu P} \text{ and } \frac{NPQ}{NP - (NP + N\varepsilon)P} = \frac{Q}{Q - \varepsilon}
$$

So
$$
\frac{T_{\lambda}}{T_{\mu}}
$$
 is greater than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.

So by making N sufficiently large, $\frac{T_{\lambda}}{T_{\lambda}}$ can be made T_{μ} λ can be mode μ greater than $\frac{1}{n}$. η $\frac{NPQ}{NP - (NP + Ne)P} \leq \frac{\lambda Q}{NP - \mu P}$ and $\frac{NPQ}{NP - (NP + Ne)P} = \frac{Q}{Q - \varepsilon}$

So $\frac{T_{\lambda}}{T_{\mu}}$ is greater than the smaller of $\left(\frac{P + \varepsilon}{P}\right)^{N+1}$ or $\left(\frac{Q}{Q - \varepsilon}\right)^{N+1}$.

So by making N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ $\leq \frac{Z}{NP - \mu P}$ and $\frac{Z}{NP - (NP + Ne)P} = \frac{Z}{Q - \varepsilon}$

er than the smaller of $\left(\frac{P + \varepsilon}{P}\right)^{N_{\varepsilon-1}}$ or $\left(\frac{Q}{Q - \varepsilon}\right)^{N_{\varepsilon-1}}$.

N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{N! P^{\kappa} Q^{N - \kappa}}{K!(N - K)!}$

1 $\frac{\lambda Q}{NP - \mu P}$ and $\frac{NPQ}{NP - (NP + N\varepsilon)P} = \frac{Q}{Q - \varepsilon}$

r than the smaller of $\left(\frac{P + \varepsilon}{P}\right)^{N\varepsilon - 1}$ or $\left(\frac{Q}{Q - \varepsilon}\right)^{N\varepsilon - 1}$.

N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{IP^{k}Q^{N-k}}{N(N - K)!}$
 $\frac{(N - \lambda +$ Eventy and $\frac{Z}{NP - \mu P}$ and $\frac{Z}{NP - (NP + N\varepsilon)P} = \frac{Z}{Q - \varepsilon}$

r than the smaller of $\left(\frac{P + \varepsilon}{P}\right)^{N\varepsilon - 1}$ or $\left(\frac{Q}{Q - \varepsilon}\right)^{N\varepsilon - 1}$.

N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{P^{\kappa}Q^{N - \kappa}}{I(N - K)!$ So $\frac{T_{\lambda}}{T_{\mu}}$ is greater than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N_{\varepsilon-1}}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N_{\varepsilon-1}}$.

So by making N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made

greater than $\frac{1}{\eta}$.

Using P(K) = $\frac{N! P$ reater than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.

ing N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{1}{\eta}$.
 $=\frac{N! P^k Q^{N-k}}{K!(N-K)!}$
 $\frac{(N-1)...(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(\lambda-1)(\lambda-2)...1}$ r than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.

N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{P^{k}Q^{N-k}}{N(N-k)!}$
 $\frac{(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(\lambda-1)(\lambda-2)...1}$
 $\frac{(N-k+1)P^{k}Q^{N-k}}{(\lambda-1)P^{\$ ter than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.

g N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{N! P^{K} Q^{N-K}}{K!(N-K)!}$
 $\frac{-1)...(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(\lambda-1)(\lambda-2)...1}$ han the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.
sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{\kappa Q^{N-K}}{N-K}$
 $\frac{(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{-1)(\lambda-2)...1}$ er than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.

N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{N! P^k Q^{N-k}}{K!(N-K)!}$
 $\frac{1)...(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(\lambda-1)(\lambda-2)...1}$ 1 2 1 the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.
ciently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{\kappa}{(1-\varepsilon)^{1/2}}$
 $\frac{\lambda+2P^{\lambda-1}Q^{N-\lambda+1}}{(-2)\cdots1}$ than the smaller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.

I sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{P^{K}Q^{N-K}}{(N-K)!}$
 $\frac{N(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{\lambda-1)(\lambda-2)...1}$ naller of $\left(\frac{P+\varepsilon}{P}\right)^{N\varepsilon-1}$ or $\left(\frac{Q}{Q-\varepsilon}\right)^{N\varepsilon-1}$.
y large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made by making N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\sum_{r=1}^{T} P(K) = \frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$
 $\sum_{r=1}^{N} = \frac{N(N-1)...(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(A-1)(A-2)...1}$
 $= \frac{N(N-1)...(N-k+1)P^{\lambda}Q^{N-k}}{k(k-1)...1}$ sufficiently large, $\frac{I_{\lambda}}{T_{\mu}}$ can be made
 $\frac{K Q^{N-K}}{N-K}$
 $\frac{(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(-1)(\lambda-2)...1}$
 $\frac{N}{K(K-1)...1}$ ing N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{1}{\eta}$.
 $= \frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$
 $\frac{(N-1)...(N-\lambda+2)P^{\lambda-1} Q^{N-\lambda+1}}{(\lambda-1)(\lambda-2)...1}$
 $\frac{(N-1)...(N-k+1)P^{k} Q^{N-k}}{k(k-1)...1}$ ufficiently large, $\frac{I_{\lambda}}{T_{\mu}}$ can be made
 $\frac{(Q^{N-k})}{(-K)!}$
 $\frac{(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{-1!(\lambda-2)...1}$
 $\frac{(N-k+1)P^{k}Q^{N-k}}{(k-1)...1}$ 7. N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{N! P^{k} Q^{N-k}}{K!(N-K)!}$
 $\frac{-1)...(N - \lambda + 2) P^{\lambda-1} Q^{N-\lambda+1}}{(\lambda - 1)(\lambda - 2)...1}$
 $\frac{-1)...(N - k + 1) P^{k} Q^{N-k}}{k(k-1)...1}$ N sufficiently large, $\frac{T_{\lambda}}{T_{\mu}}$ can be made
 $\frac{\sqrt{1 P^{K} Q^{N-K}}}{\sqrt{1 (N-K)!}}$
 $\frac{(1 - \lambda)^{N-K} (N - \lambda + 2) P^{\lambda - 1} Q^{N - \lambda + 1}}{(2 - 1)(\lambda - 2)...1}$
 $\frac{1}{2}$
 $\frac{(N - k + 1) P^{k} Q^{N-k}}{(k - 1)...1}$ ciently large, $\frac{I_{\lambda}}{T_{\mu}}$ can be made
 $\frac{k}{(n+1)!}$
 $\frac{(k+2)P^{\lambda-1}Q^{N-\lambda+1}}{(-2)...1}$
 $\frac{-k+1)P^kQ^{N-k}}{(-2)!}$

Using $P(K) = \frac{N! P^{K} Q^{N-K}}{K! (N-K)!}$

$$
T_{\lambda-1} = \frac{N(N-1)...(N-\lambda+2)P^{\lambda-1}Q^{N-\lambda+1}}{(\lambda-1)(\lambda-2)...1}
$$

$$
T_k = \frac{N(N-1)...(N-k+1)P^kQ^{N-k}}{k(k-1)...1}
$$

So
$$
\frac{T_{\lambda-1}}{T_k} = \frac{(N-k)(N-k-1)...(N-\lambda+2)P^{\lambda-k-1}}{(\lambda-1)(\lambda-2)...(k+1)Q^{\lambda-k-1}}
$$

Reversing the order of the factors in both the numerator and

Reversing the order of the factors in both the numerator and denominator , so that the factors will be increasing instead of decreasing gives: $P^{\lambda-k-1}$
th the numerator and
increasing instead of
 $\frac{N-k}{\lambda-1} * \frac{P^{\lambda-k-1}}{Q^{\lambda-k-1}}$
* $NP - kP$ erator and
instead of
 $\frac{-k-1}{-k-1}$ erator and
instead of
 $\frac{-k-1}{-k-1}$ numerator and

asing instead of

* $\frac{P^{\lambda-k-1}}{Q^{\lambda-k-1}}$
 $\frac{P-kP}{Q-Q}$

e previous fraction by

minator. The first

So
$$
\frac{T_{\lambda-1}}{T_k} = \frac{(N-k)(N-k-1)...(N-\lambda+2)P^{\lambda-k-1}}{(\lambda-1)(\lambda-2)...(k+1)Q^{\lambda-k-1}}
$$

Reversing the order of the factors in both the numerator and denominator, so that the factors will be increasing instead of decreasing gives:

$$
\frac{T_{\lambda-1}}{T_k} = \frac{N-\lambda+2}{k+1} * \frac{N-\lambda+3}{k+2} ** \frac{N-k}{\lambda-1} * \frac{P^{\lambda-k-1}}{Q^{\lambda-k-1}}
$$

So
$$
\frac{T_{\lambda-1}}{T_k} = \frac{NP-\lambda P+2P}{kQ+Q} * \frac{NP-\lambda P+3P}{kQ+2Q} ** \frac{NP-kP}{\lambda Q-Q}
$$

Notice that each fraction is obtained from the previous fraction by adding P to the numerator and Q to the denominator. The first

 Notice that each fraction is obtained from the previous fraction by adding P to the numerator and Q to the denominator . The first Reversing the order of the factors in both the numerator and
denominator, so that the factors will be increasing instead of
decreasing gives:
 $\frac{T_{\lambda-1}}{T_k} = \frac{N - \lambda + 2}{k + 1} * \frac{N - \lambda + 3}{k + 2} * ... * \frac{N - k}{\lambda - 1} * \frac{P^{\lambda - k-1}}{Q$ kQ and \sim \sim \sim \sim $-AP+P$ by adding D to the numerator and Q to the denominator. There are λ -k-1 fractions in the product, so by the same reasoning as in lemma 7, $rac{\frac{f_{\lambda+1}}{f_k}}{\frac{f_{\lambda+1}}{f_k}} = \frac{N - \lambda + 2}{k + 1} * \frac{N - \lambda + 3}{k + 2} * * \frac{N - \kappa}{\lambda - 1} * \frac{P}{Q^{\lambda - k - 1}}$
 $\frac{\frac{f_{\lambda+1}}{f_k}}{\frac{N}{f_k}} = \frac{NP - \lambda P + 2P}{kQ + Q} * \frac{NP - \lambda P + 3P}{kQ + 2Q} * * \frac{NP - kP}{\lambda Q - Q}$

Notice that each fraction is obtain ... * $\frac{N-k}{\lambda-1}$ * $\frac{P^{\lambda-k-1}}{Q^{\lambda-k-1}}$

3P * * $\frac{NP-kP}{\lambda Q-Q}$

ed from the previous fraction by

b the denominator . The first
 $\frac{\lambda-AP+P}{kQ}$ by adding P to the

or. There are $\lambda-k-1$ fractions

oning as in le $\frac{N-K}{\lambda-1} * \frac{P}{Q^{\lambda-k-1}}$

....* $\frac{NP-kP}{\lambda Q-Q}$

om the previous fraction by

denominator . The first
 $\frac{P}{\lambda}$ by adding P to the

nere are $\lambda-k-1$ fractions

g as in lemma 7,
 $\frac{\lambda P+P}{Q}$) $\lambda-k-1$ or $\left(\frac{NP-kP}{(\lambda Q-Q)}$ Fraction by

e first

o the

Fractions
 $\left(\frac{P-kP}{Q-Q}\right)^{\lambda-k-1}$.

his is the
 $NP - kP$

NP- kP fraction by

The first

to the

fractions

,
 $\frac{NP - kP}{(\lambda Q - Q)}$
 λ this is the
 $(NP - kP)$ $Nc-1$

T_k kQ + *Q kQ* + 2*Q λQ* − *Q*
\nNotice that each fraction is obtained from the previous fraction by
\nadding P to the numerator and Q to the denominator. The first
\nfraction is itself obtained from
$$
\frac{NP - λP + P}{kQ}
$$
 by adding P to the
\nnumerator and Q to the denominator. There are λ-k-1 fractions
\nin the product, so by the same reasoning as in lemma 7,
\n $\frac{T_{\lambda-1}}{T_k}$ is greater than the smaller of $(\frac{NP - λP + P}{kQ})^{\lambda-k-1}$ or $(\frac{NP - kP}{(λQ - Q)})^{\lambda-k-1}$.
\nIf NP and Nε are both integers then λ-k-1 = Nε -1 and this is the
\nsmallest value that λ-k-1 can have.
\nSo $\frac{T_{\lambda-1}}{T_k}$ is greater than the smaller of $(\frac{NP - λP + P}{kQ})^{Nε-1}$ or $(\frac{NP - kP}{(λQ - Q)})^{Nε-1}$
\n $\frac{NP - λP + P}{kQ} > \frac{NP - (NP + 1)P + P}{(NP - Nε)Q} = \frac{NP(1 - P)}{(NP - Nε)(1 - P)} = \frac{P}{P - ε}$.

$$
\frac{NP - kP}{\lambda Q - Q} > \frac{NP - (NP - N\varepsilon)P}{(NP + 1)Q - Q} = \frac{NP(1 - P) + N\varepsilon P}{NPQ} = \frac{Q + \varepsilon}{Q}.
$$

So
$$
\frac{T_{\lambda-1}}{T_k}
$$
 is greater than the smaller of $\left(\frac{P}{P-\varepsilon}\right)^{N\varepsilon-1}$ or $\left(\frac{Q+\varepsilon}{Q}\right)^{N\varepsilon-1}$.

So by making N sufficiently large, $\frac{T_{\lambda-1}}{T}$ can be made greater than T_k and ϵ λ -1 can he made oreater th $\frac{1}{\eta}$. η

So if N is sufficiently large $\frac{T_{\lambda}}{T} > \frac{1}{2}$ and $\frac{T_{\lambda-1}}{T} > \frac{1}{2}$. T_{μ} η T_{k} η $\lambda \sim 1$ and $\lambda - 1 \sim 1$ μ μ μ τ 1 and $T_{\lambda-1} > 1$ η I_k η $T_{\lambda-1}$ 1 T_k η $\frac{1}{\lambda-1}$ $\frac{1}{\lambda}$ η So $\frac{T_{\mu}}{T} < \eta$ and $\frac{T_{k}}{T} < \eta$. T_{λ} is the set of $T_{\lambda-1}$ μ and λ and λ λ λ λ T_k \geq 32 $T_{\lambda-1}$ $k \geq n$ $\lambda - 1$

So if N is sufficiently large, $C < A \left(\frac{\eta}{1 - \eta} \right)$ and $D < B \frac{\eta}{1 - \eta}$ $1-\eta$) $1-\eta$ $\left(\eta\right)$ and $D < 1$ $\left(\frac{7}{1-\eta}\right)$ and $D \leq r$) and $D < B$ η) and $D \leq B \frac{\eta}{1-\eta}$ $1-\eta$

This completes the proof

Daniel Daniels